M923: How to Win at (Some) Games

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Winter 2022

What is game theory?

- **Game theory** is "the study of mathematical models of conflict and cooperation between intelligent rational decision-makers." [3]
  - **Economonic games** usually have players that move simultaneously and result in gains or losses. [1]
    Examples:
    - the prisoner's dilemma
    - predator-prey relationships
    - the stock market
  - **Combinatorial games** usually have players that move sequentially and have perfect information, and result in one player winning.
    Examples:
    - tic-tac-toe
    - chess
    - go

Impartial Games

An **impartial game** [2] is a special kind of game, in which:

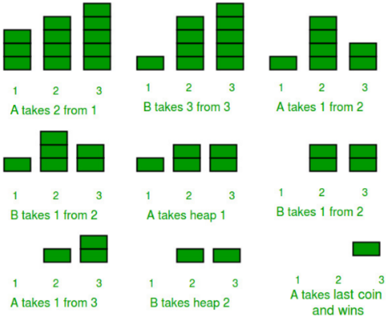
1. There are only two players.
2. There are several **positions**, and often a particular starting position.
3. There are rules that specify which moves either player can make from the current position to its **options**.
4. Players alternate taking turns.
5. Both players know what is going on, i.e. there is **complete information**.
6. The last player to move wins (the **normal play** rule).
   If the last player to move loses, we are using the **misére play** rule.
7. There are no **chance moves** such as rolling dice or shuffling cards.
8. The game cannot go on forever (e.g. by repeating the same moves over and over again).
9. Both players always have the same moves available to them.
   If each player has different options, we call the game **partizan**.

Example: Chess

	Chess
Two players	
Positions	
Rules	
Alternating moves	
Complete information	
Normal play	
No chance moves	
Ending condition	
Same moves for both players	

How to play Nim

- There are two players.
- The game is played with a bunch of piles of chips.
- Players take turns choosing one pile, and taking one or more chips from that pile.
- The winner is the player who takes the last chip from the last remaining pile.



How to play Notakto

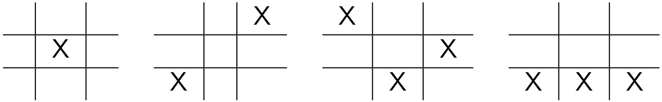
- There are two players.
- The game is played on a normal tic-tac-toe board.
- Players take turns placing X's on the board until three are placed in a row.
- The player who makes 3-in-a-row wins.

Notation

Definition

Given a position, if the *next* player to make a move has a winning strategy, then we call it an $\mathcal{N}$-**position**. Otherwise, the *previous* player must have a winning strategy, so we call it a $\mathcal{P}$-**position**.

Question: Classify these Notakto positions.



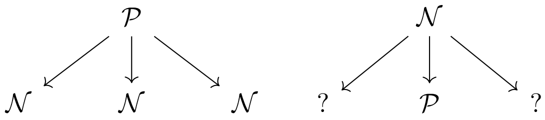
Games on Chessboards

- The game is played on an infinite chessboard with a star at a corner.
- A rook is placed somewhere on the board, and players take turns moving the rook closer to the star.
- The player who moves the rook to the star wins.

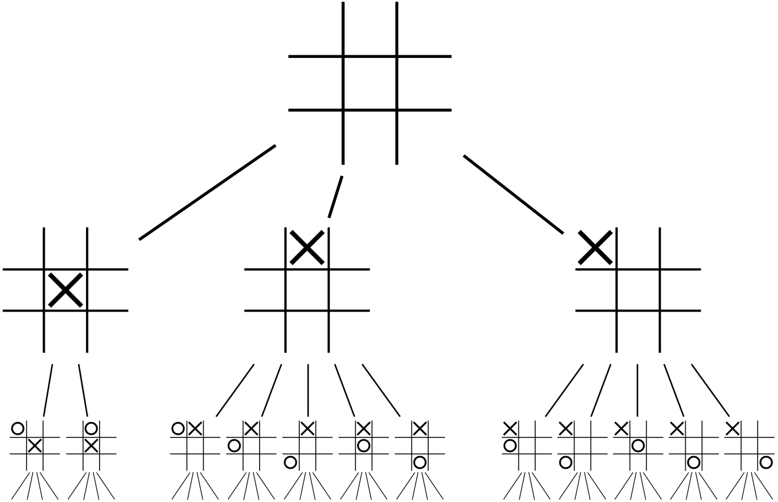
$\mathcal{P}$- and $\mathcal{N}$-positions

Lemma

1. *A game is a $\mathcal{P}$-position if all the options are $\mathcal{N}$-positions.*
2. *A game is an $\mathcal{N}$-position if at least one option is a $\mathcal{P}$-position.*



Game Trees



Analyzing Game Trees: Part 1

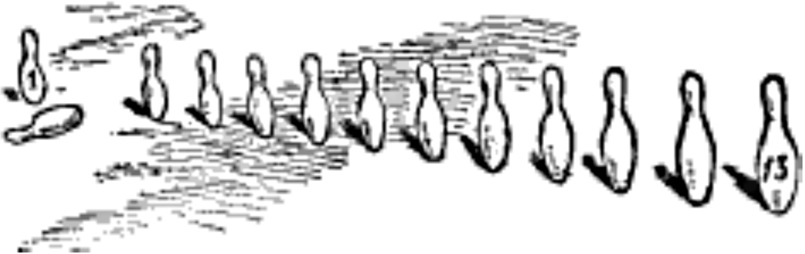
Idea: Label the vertices of the game tree with $\mathcal{P}$ and $\mathcal{N}$.
We can follow this simple algorithm to figure out who wins a game based on its game tree:

1. If any vertex has no outgoing edges (so, a position with no options), label it with a $\mathcal{P}$.
2. If any vertex points to a $\mathcal{P}$, label it with an $\mathcal{N}$. If any vertex points only to $\mathcal{N}$s, label it with a $\mathcal{P}$.
3. Repeat step 2 until all vertices are labeled.

Pop Quiz

The two-player game of **Kayles** is played with a bowling ball and a row of 13 bowling pins. Players take turns rolling the ball to either hit one pin, or two adjacent pins. The winner is the player who knocks over the last pin.

Which player has a winning strategy?
Does it depend on the number of pins?



Breaking up Nim

Idea: Break up a game of Nim into smaller games of Nim.

Definition

In the **sum of two games**, players take turns moving in *one game at a time*. Once a game is won, players are not allowed to move in that game anymore. The winner is the player who wins the last game.

Nimbers

Problem: $\mathcal{P}$- and $\mathcal{N}$-positions don't tell us enough information to figure out who wins a sum of Nim-heaps.
Solution: Forget $\mathcal{P}$- and $\mathcal{N}$-positions. Keep track of the number of chips instead.

Definition

The **nimber** n is a nim-heap with n chips.

Nim Sum

Definition

The **minimum excluded value**, or **mex**, of a set of natural numbers is the smallest natural number not in the set.

Theorem

The sum of the nimbers x and y is the smallest nimber that isn't an option of $x + y$. In other words, the mex of the options of $x + y$.

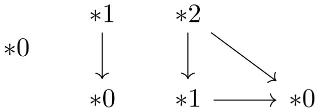
Experiment: Fill out the first few cells of the nimber addition table. Do you notice any patterns?

How to win Nim

1. Break the game up into Nim-heaps.
2. Count the number of chips in each heap.
3. Use nimber addition to combine the values of heaps two at a time.
4. Check that the nim-value of the game is *not* zero.
5. Make a move to any zero position.

Analyzing Game Trees: Part 2

Idea: Label the vertices of the game tree with nimbers.



We can label any game tree as follows:

1. If any vertex has no outgoing edges (so, a position with no options), label it with a $*0$.
2. If any vertex points to only already-labeled vertices, label it with the mex of its options.
3. Repeat step 2 until all vertices are labeled.

The Sprague-Grundy Theorem

Theorem (Sprague-Grundy)

Under the normal play convention, any impartial game is equivalent to a nimber, called the *Grundy value* or *nim-value* of the game.

We can compute this value via the tree-labeling algorithm described in the previous slide. The best move to make in an impartial game, then, is one that changes the value of the game to $*0$.

Corollary

Any sum of impartial games is equivalent to a game of Nim (which, then, is itself equivalent to a nimber).

How to win any impartial game

1. Break the game up into smaller subgames (if possible).
2. Use game trees to find the nim-value of each subgame.
3. Use nimber addition to combine the values of the subgames two at a time.
4. Check that the nim-value of the game is *not* zero.
5. Make a move to any zero position.

Applications

Impartial games have been used in cryptography to create "error-correcting codes".

IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. IT-32, NO. 3, MAY 1986 337

Lexicographic Codes: Error-Correcting Codes from Game Theory

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Abstract—Lexicographic codes, or lexicodes, are defined by various versions of the greedy algorithm. The theory of these codes is closely related to the theory of certain impartial games, which leads to a number of surprising properties. For example, lexicodes over an alphabet of size $B = 2^a$ are closed under addition, while if $B = 2^{2^a}$ the lexicodes are closed under multiplication by scalars, where addition and multiplication are in the nim sense explained in the text. Hamming codes and the binary Golay codes are lexicodes. Remarkably simple constructions are given for the Steiner systems $S(5,6,12)$ and $S(5,8,24)$. Several record-breaking constant weight codes are also constructed.

I. Introduction

THIS PAPER is concerned with various classes of lexicographic codes, that is, codes that are defined by a greedy algorithm: each successive codeword is selected as the *first* word not prohibitively near (in some prescribed sense) to earlier codewords. For example, the very simplest class of lexicographic codes is defined as follows. We specify a base B and a desired minimal Hamming distance d. The first codeword accepted is the zero word. Then we consider all base-B vectors in turn, and accept a vector as a codeword if it is at Hamming distance at least d from all previously accepted codewords. (An example with $B = 3$ and $d = 3$ can be seen in Table XI.)

One of our goals is to point out the essential identity between this kind of lexicographic coding theory and the theory of certain impartial games (see Section II). Then the Sprague–Grundy theory of games has a number of interesting and surprising consequences for lexicographic codes (or *lexicodes*).

1) Unrestricted binary lexicodes are linear (Theorems 1, 3).

2) For base $B = 2^a$, unrestricted lexicodes are closed under nim-addition (Theorem 4).

3) For base $B = 2^{2^a}$, unrestricted lexicodes are closed under nim-multiplication, which is an operation that converts the digits $\{0, 1, 2, 3, \cdots, 2^{2^a} - 1\}$ into a field (Theorem 5).

Two other results worth mentioning here are the following.

5) Several well-known codes unexpectedly turn out to be lexicographic codes, including Hamming codes and the binary Golay codes of length 23 and 24 (Section III-B).

6) The constant weight binary lexicode of length 24, distance 8 and weight 8 is the Steiner system $S(5,8,24)$ (Theorem 12). By imposing an additional constraint on a constant weight lexicode (see Section IV-E), Ryba obtained an almost equally simple construction for the Steiner system $S(5,6,12)$ (Theorem 13). The corresponding game, called Mathematical Blackjack (or Mathieu's Vingt-et-un) is described at the end of Section IV-E.

7) A number of constant weight codes with minimal distance 10 and containing a record number of codewords are given in Table XIII.

Some of the game-theoretic aspects of this work are described in [1] and [2]. The relations between the theories of games and of lexicographic codes, and in particular the multiplicative theorem, underly some of the results in [1]. However, most of the results are published here for the first time. This work may be regarded as a coding-theoretic analog of the laminated lattices described in [5], [6].

The paper is arranged as follows. The connections with game theory are discussed in Section II, unrestricted lexicodes are treated in Section III, and Section IV deals with constant weight and constrained lexicodes. Tables IV–VIII and XII give the parameters of a number of lexicodes.

II. The Connections with Game Theory

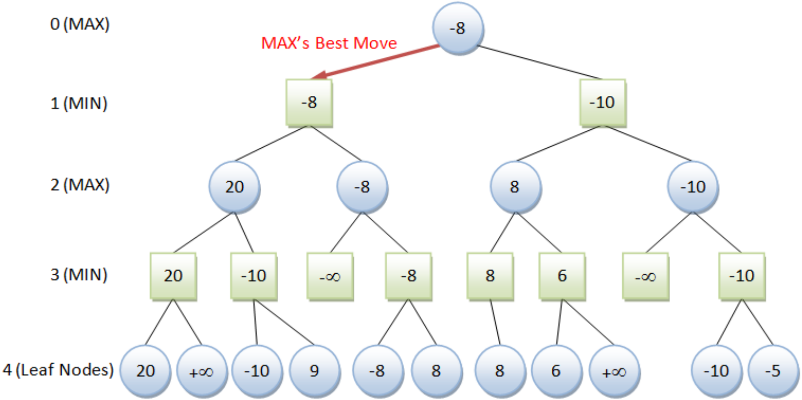
A. Grundy's Game

We begin by describing Grundy's game [1, p. 96], [9, p. 8], which is a characteristic example of the class of games to be considered. In Grundy's game the typical position

$$P_a + P_b + P_c + \cdots \quad (1)$$

Applications

Game trees are a kind of decision tree, which are used in computer science to build and study artificial intelligence.



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