How to Win at (Some) Games

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    Examples:
    - the prisoner's dilemma
    - predator-prey relationships
    - the stock market
  - **Combinatorial games** usually have players that move sequentially and have perfect information, and result in one player winning.
    Examples:
    - tic-tac-toe
    - chess
    - go

Outline

- What are impartial games?
- How do we play?
- How do we analyze?
- How do we win?

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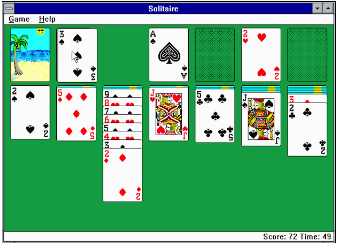
   If Left and Right have different options, we call the game **partizan**.

Examples 1

	Solitaire
Two players	
Positions	
Rules	
Alternating moves	
Complete information	
Normal play	
No chance moves	
Ending condition	
Same moves for both players	



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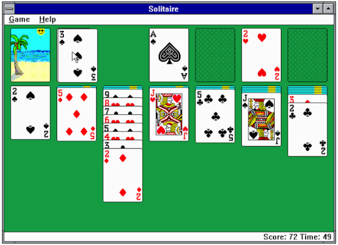

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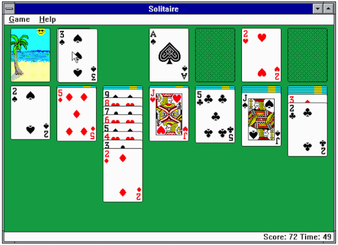

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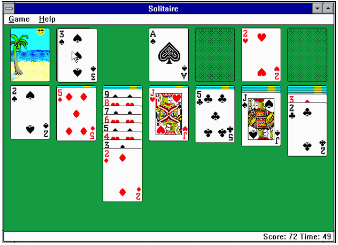

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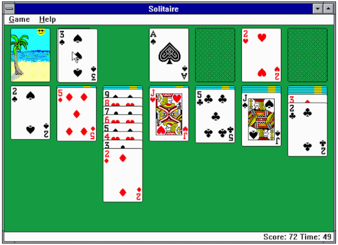

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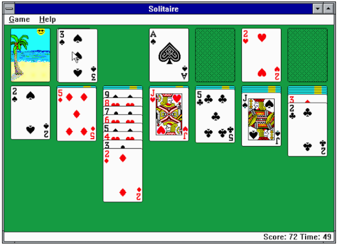

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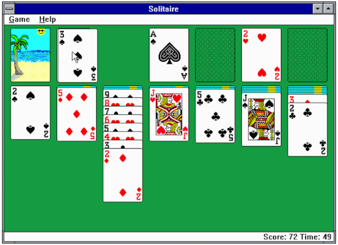

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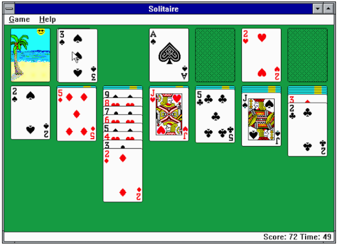

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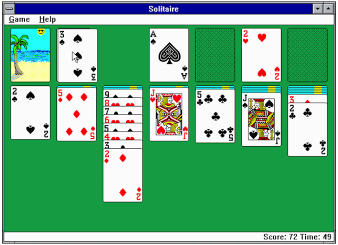

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Complete information	✗
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Ending condition	✗
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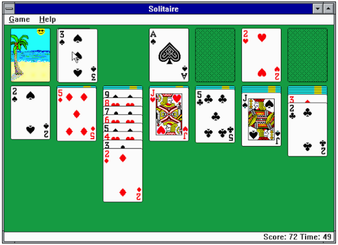

Examples 2

	Tic-tac-toe	Chess	Battleship	Monopoly
Two players	✓	✓	✓	?
Positions	✓	✓	✓	✓
Rules	✓	✓	✓	✓
Alternating moves	✓	✓	✓	?
Complete information	✓	✓	✗	✗
Normal play	✗	✗	✓	✗
No chance moves	✓	✓	✓	✗
Ending condition	✓	✗	✓	✗
Same moves for both players	✗	✗	✗	✗

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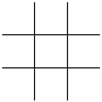

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- There are two players, Left and Right.
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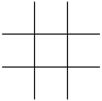
- Players take turns placing X's on the board until three are placed in a row.



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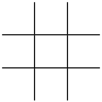

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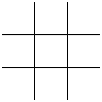

Do This: Play a couple rounds of Notakto.

Notation

Definition

Given a position, if the *next* player to make a move has a winning strategy, then we call it an $\mathcal{N}$-**position**. Otherwise, the *previous* player must have a winning strategy, so we call it a $\mathcal{P}$-**position**.

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Question: Classify these Notakto positions.

	X	

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X		

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X	X	X

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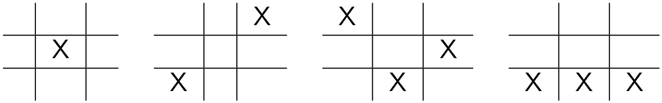

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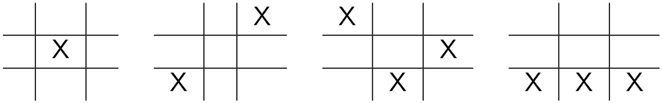

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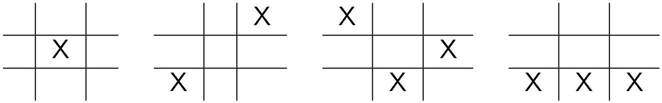

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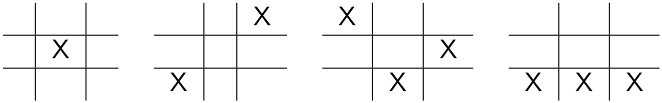

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Games on Chessboards

- The game is played on an infinite chessboard with a star at a corner.

$\cdots$					$\star$
$\cdots$					
$\cdots$					
$\cdots$					
$\cdots$					
$\iddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

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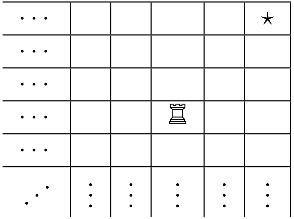
- A rook is placed somewhere on the board, and players take turns moving the rook closer to the star.



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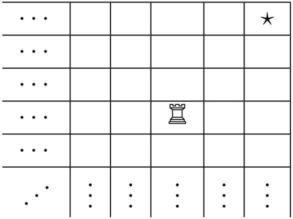
- The player who moves the rook to the star wins!



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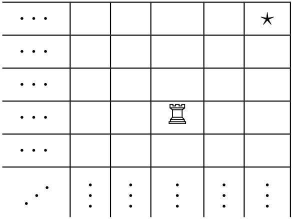
Do This: Play a couple rounds of the rook game.



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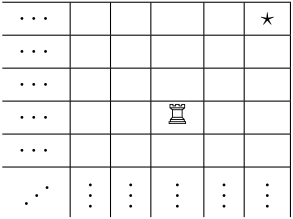
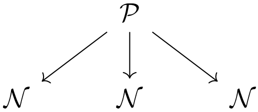
Question: Which spaces are $\mathcal{P}$- and $\mathcal{N}$-positions?



$\mathcal{P}$- and $\mathcal{N}$-positions

Lemma

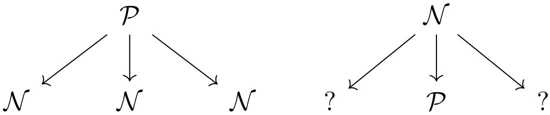
1. *All of the options from a $\mathcal{P}$-position are $\mathcal{N}$-positions.*



$\mathcal{P}$- and $\mathcal{N}$-positions

Lemma

1. *All of the options from a $\mathcal{P}$-position are $\mathcal{N}$-positions.*
2. *At least one of the options from an $\mathcal{N}$-position is a $\mathcal{P}$-position.*



Adding Games

Definition

In the **sum of two games**, players take turns moving in *one game at a time*. Once a game is won, players are not allowed to move in that game anymore. The winner is the player who wins the last game.

e.g. Notakto + Notakto

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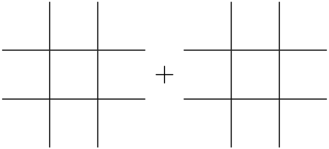

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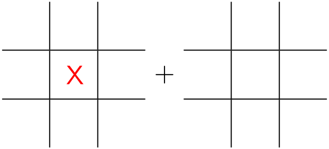

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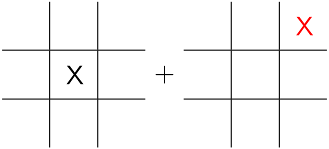

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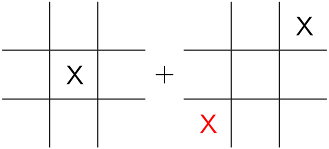

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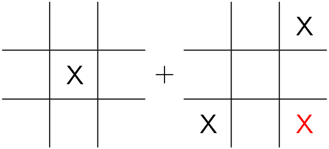

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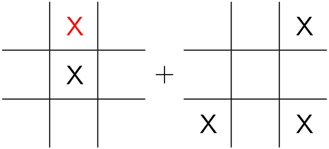

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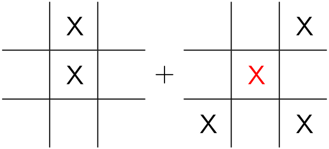

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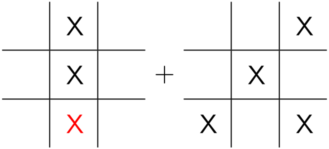

Player 1 wins!

Adding Games

Definition

In the **sum of two games**, players take turns moving in *one game at a time*. Once a game is won, players are not allowed to move in that game anymore. The winner is the player who wins the last game.

e.g. Notakto + Notakto

	X	
	X	
	X	

$+$

		X
	X	
X		X

Player 1 wins!

Do This: Practice playing Notakto+Notakto.

Question: How do $\mathcal{N}$- and $\mathcal{P}$-positions add up?

$+$	$\mathcal{P}$	$\mathcal{N}$
$\mathcal{P}$		
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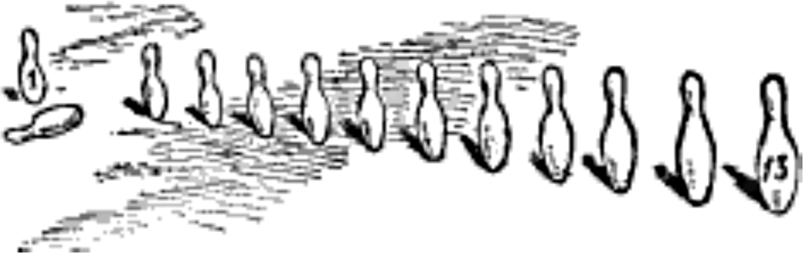
Conclusion: $\mathcal{P}$- and $\mathcal{N}$-positions don't give us enough information.
Goal: Find a better way to categorize impartial games!

Break

Let's take a 5 minute break.

Pop Quiz

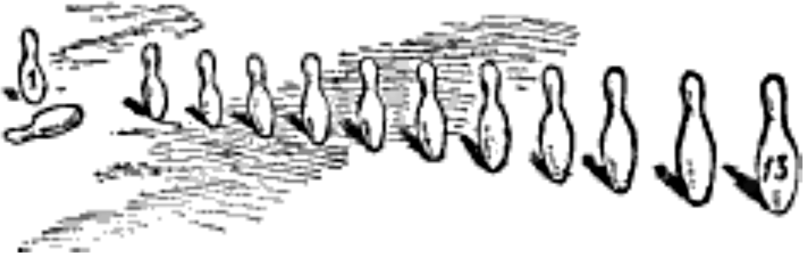
The two-player game of **Kayles** is played with a bowling ball and a row of 13 bowling pins. Players take turns rolling the ball to either hit one pin, or two adjacent pins. The winner is the player who knocks over the last pin.



Pop Quiz

The two-player game of **Kayles** is played with a bowling ball and a row of 13 bowling pins. Players take turns rolling the ball to either hit one pin, or two adjacent pins. The winner is the player who knocks over the last pin.

Which player has a winning strategy?
Does it depend on the number of pins?



Overview

Goal

Come up with a system for analyzing any impartial game under normal play.

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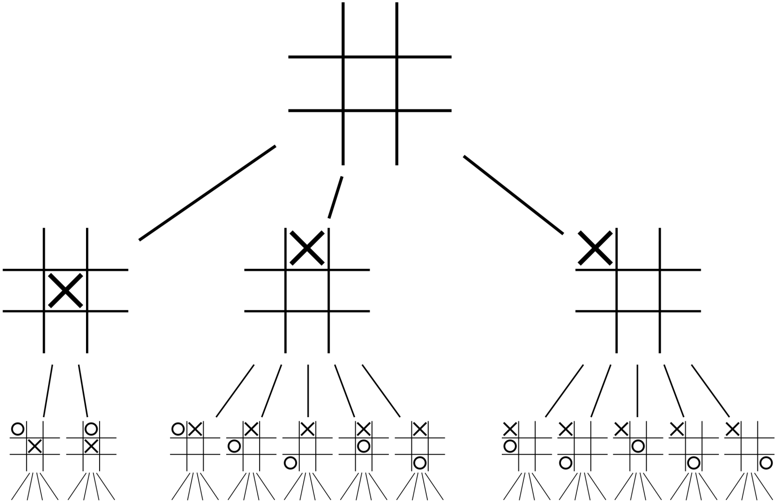
Come up with a system for analyzing any impartial game under normal play.

Strategy: Divide-and-conquer!

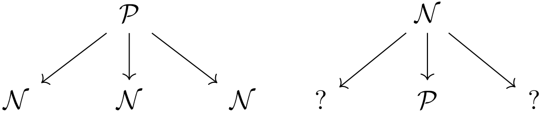
1. Split the game up into a sum of smaller subgames.
2. Analyze the subgames.
3. Combine the subgames, two-at-a-time, until we have a solution for the whole game.

Game Trees



Analyzing Game Trees

Remember our rules for $\mathcal{P}$- and $\mathcal{N}$-positions?



We can use them to label any game tree as follows:

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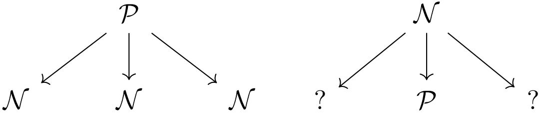

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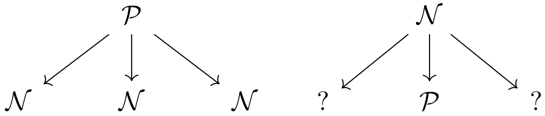

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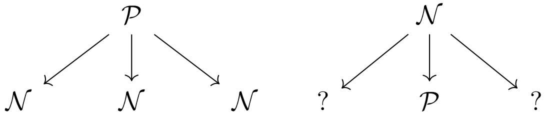

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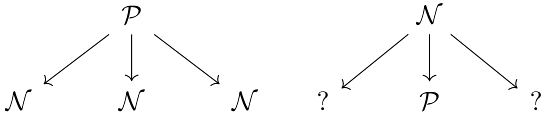

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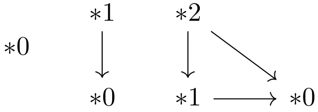
4. Repeat steps 2 and 3 until all vertices are labeled.

Analyzing Game Trees

Idea: Instead of labeling vertices with $\mathcal{N}$ and $\mathcal{P}$, we can label them with nimbers!



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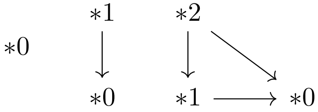

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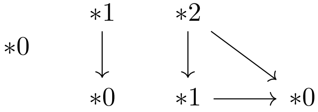

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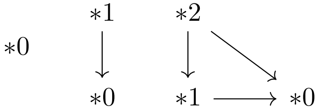

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The Sprague-Grundy Theorem

Theorem (Sprague-Grundy)

Under the normal play convention, any impartial game is equivalent to a nimber, called the *Grundy value* or *nim-value* of the game.

We can compute this value via the tree-labeling algorithm described in the previous slide. The best move to make in an impartial game, then, is one that changes the value of the game to $*0$.

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Corollary

Any sum of impartial games is equivalent to a game of Nim (which, then, is itself equivalent to a nimber).

How does this help me win at Nim?

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- Find the nim-values of the Nim games with the following piles:
    - $\{2, 3\}$
    - $\{3, 5\}$
    - $\{1, 2, 3, 4\}$
    - $\{1, 3, 57, 2, 1, 7, 57\}$
    - $\{6, 8, 5, 3\}$
- (Problem 3) Fill as much of the Nim addition table as you need.

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Applications

Impartial games have been used in cryptography to create "error-correcting codes".

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Lexicographic Codes: Error-Correcting Codes from Game Theory

JOHN H. CONWAY AND N. J. A. SLOANE, FELLOW, IEEE

Abstract—Lexicographic codes, or lexicodes, are defined by various versions of the greedy algorithm. The theory of these codes is closely related to the theory of certain impartial games, which leads to a number of surprising properties. For example, lexicodes over an alphabet of size $B = 2^a$ are closed under addition, while if $B = 2^{2^a}$ the lexicodes are closed under multiplication by scalars, where addition and multiplication are in the nim sense explained in the text. Hamming codes and the binary Golay codes are lexicodes. Remarkably simple constructions are given for the Steiner systems $S(5,6,12)$ and $S(5,8,24)$. Several record-breaking constant weight codes are also constructed.

I. Introduction

THIS PAPER is concerned with various classes of lexicographic codes, that is, codes that are defined by a greedy algorithm: each successive codeword is selected as the *first* word not prohibitively near (in some prescribed sense) to earlier codewords. For example, the very simplest class of lexicographic codes is defined as follows. We specify a base B and a desired minimal Hamming distance d. The first codeword accepted is the zero word. Then we consider all base-B vectors in turn, and accept a vector as a codeword if it is at Hamming distance at least d from all previously accepted codewords. (An example with $B = 3$ and $d = 3$ can be seen in Table XI.)

One of our goals is to point out the essential identity between this kind of lexicographic coding theory and the theory of certain impartial games (see Section II). Then the Sprague–Grundy theory of games has a number of interesting and surprising consequences for lexicographic codes (or *lexicodes*).

1) Unrestricted binary lexicodes are linear (Theorems 1, 3).

2) For base $B = 2^a$, unrestricted lexicodes are closed under nim-addition (Theorem 4).

3) For base $B = 2^{2^a}$, unrestricted lexicodes are closed under nim-multiplication, which is an operation that converts the digits $\{0, 1, 2, 3, \cdots, 2^{2^a} - 1\}$ into a field (Theorem 5).

4) The constant weight binary lexicodes with minimal distance 4 have a rather subtle complete solution in terms

Two other results worth mentioning here are the following.

5) Several well-known codes unexpectedly turn out to be lexicographic codes, including Hamming codes and the binary Golay codes of length 23 and 24 (Section III-B).

6) The constant weight binary lexicode of length 24, distance 8 and weight 8 is the Steiner system $S(5,8,24)$ (Theorem 12). By imposing an additional constraint on a constant weight lexicode (see Section IV-E), Ryba obtained an almost equally simple construction for the Steiner system $S(5,6,12)$ (Theorem 13). The corresponding game, called Mathematical Blackjack (or Mathieu's Vingt-et-un) is described at the end of Section IV-E.

7) A number of constant weight codes with minimal distance 10 and containing a record number of codewords are given in Table XIII.

Some of the game-theoretic aspects of this work are described in [1] and [2]. The relations between the theories of games and of lexicographic codes, and in particular the multiplicative theorem, underly some of the results in [1]. However, most of the results are published here for the first time. This work may be regarded as a coding-theoretic analog of the laminated lattices described in [5], [6].

The paper is arranged as follows. The connections with game theory are discussed in Section II, unrestricted lexicodes are treated in Section III, and Section IV deals with constant weight and constrained lexicodes. Tables IV–VIII and XII give the parameters of a number of lexicodes.

II. The Connections with Game Theory

A. Grundy's Game

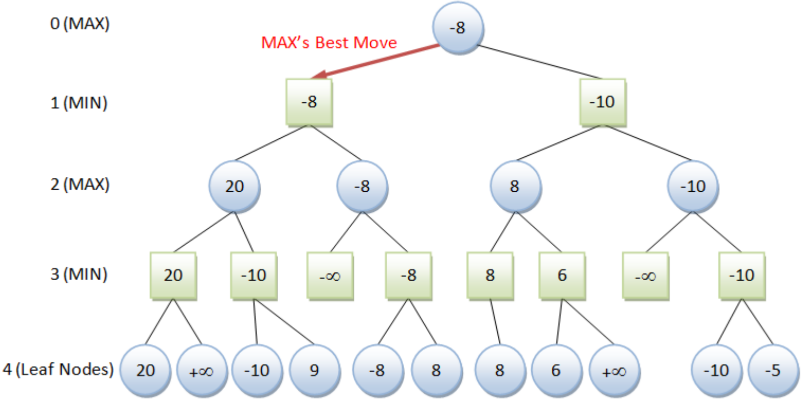
We begin by describing Grundy's game [1, p. 96], [9, p. 8], which is a characteristic example of the class of games to be considered. In Grundy's game the typical position

$$P_a + P_b + P_c + \cdots \quad (1)$$

consists of a number of heaps containing

Applications

Game trees are a kind of decision tree, which are used in computer science to build and study artificial intelligence.



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