Will the bird get home? and other stories in random walks

Rainstorm 2020

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1 What is a random walk?

What exactly is a random walk?

Well let's suppose your mom told you to go buy some groceries from the supermarket, which she tells you is down the street. But it isn't until you go outside that you realize - you don't know which way to go!

Any ordinary person would go inside and ask their mom. Phyllis is not normal and decides she will guess.

Now, let's stop for a moment. Even if she refused help from anyone, she should obviously just walk down for a while and then if she didn't see it, go the other way. But having learned about random walks, she decides to be mathematically precise and flip a coin at each step: heads, she will take a step in one direction, and tails, she will take a step in the other direction.

Obviously, this wastes a lot of time. For example she might flip HTHTHTHT, in which case she literally would not have moved at all. But on the flip side (hahaha!), she might get HHHHHHH, or TTTTTTTTTTTTTTTTTTTTTTTTTT. You never know! The fantastic thing about random walks is that they encompass a lot more possibilities than non-random walks.

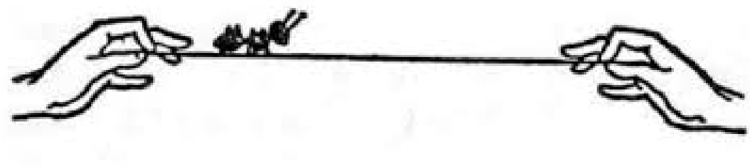
The point is exactly that: random walks are walks where each decision is based on probability, and is not determined.

Today, we'll examine the question of whether Phyllis will ever find the supermarket when:

- it's down the street "somewhere;"
- it exists somewhere on the face of the planet;
- it might be floating off in space.

2 Random walks on a 1D lattice

Imagine Phyllis as a drunk ant on a string, and somewhere on the string is food. She can't quite detect it but she wants to find the food. For mathematical purposes, this string is infinitely long. (What? Is there a problem with that?)



She decides that at every step, she will flip a coin: heads, she will take a step in one direction, and tails, she will take a step in the other direction.

I will admit that because I am evil, I did not actually put food on the string. I just told her that there is food.

Now I know she will keep searching until she finds food. I'm not *that* evil, so I put a sign back at where she started to tell her that there was no food in the first place and she should give up. But she will only find this sign if she ever returns home!

The real question is: **will she ever return home?**

(Spoiler: she will. Because otherwise her mom will get mad at her.)

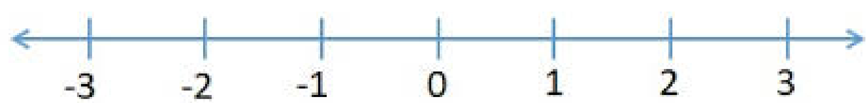
3 Lattices

We'll briefly mention what a lattice is, for the sake of clarity.

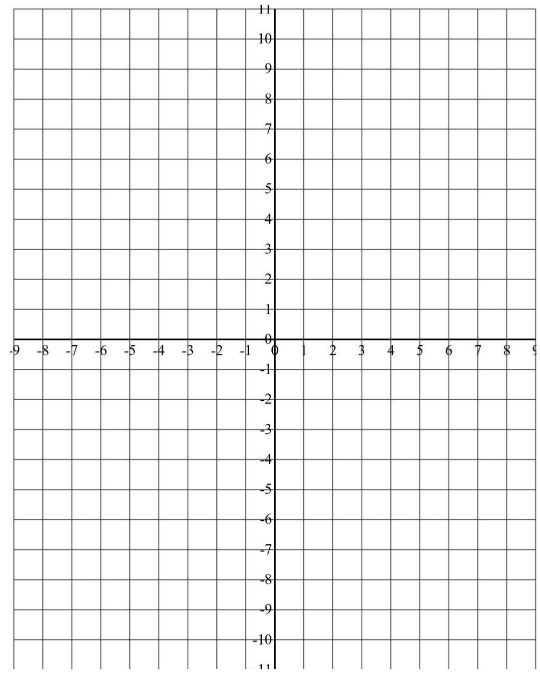
An n-dimensional **lattice** is equivalent to $\mathbb{Z}^n$: that is, n-tuples of integers under the operation of addition component-wise. ($\mathbb{Z}$ refers to all of the integers.) You can think about these as n-vectors, except their components can only be integers. You can add vectors and subtract vectors.

Why $\mathbb{Z}$, rather than (as perhaps you've seen before) $\mathbb{R}$ (th real numbers) or $\mathbb{C}$ (the complex numbers)? Well, as you may be familiar with, $\mathbb{R}^2$ is the coordinate plane, while $\mathbb{R}$ refers to the real line. These are useful for infinitesimal, or continuous, actions. On the other hand, sometimes it's simpler to think in terms of discrete actions: Phyllis's steps will all be approximately the same size. She will not be taking $1000^{1000000}$-meter steps, nor will she be taking 10^{-35}-meter steps. In $\mathbb{Z}$, every step is discrete: there is a minimal length, and there is a notion of "nearest number," which doesn't exist in $\mathbb{R}$ or $\mathbb{C}$. Every step is the same length.

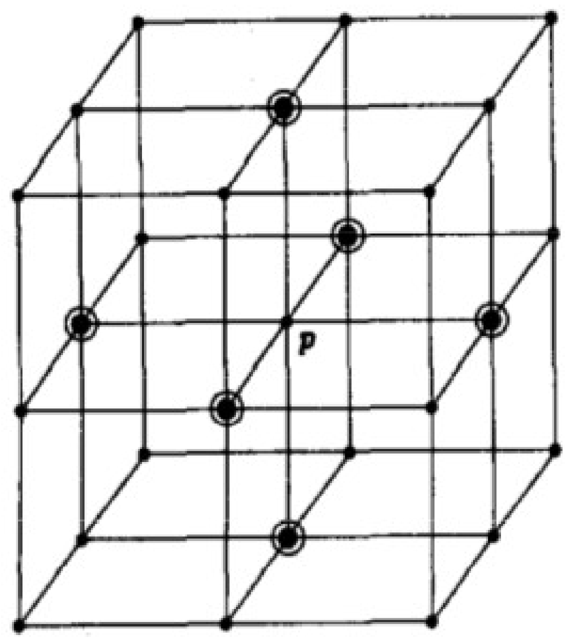
The one-dimensional lattice, $\mathbb{Z}$, looks like the following:



The two-dimensional lattice, $\mathbb{Z}^2$, looks like:



The three dimensional lattice, $\mathbb{Z}^3$, looks like:



4 An ant on a string

Now let's return to the question in section 2: Phyllis the ant walking on a string. Let's state it more precisely as follows. Phyllis is on the 1D lattice $\mathbb{Z}$. She starts at 0, and every second, she flips a coin. Supposing she is at the integer n, if she flips heads, she moves to $n + 1$, and if she flips tails, she moves to $n - 1$. Does she ever return to 0?

Let's assume for the moment that she first flips heads, and goes to 1. (If her first flip was tails, the argument is exactly the same.) We will calculate the probability that she never returns home. Then with probability $\frac{1}{2}$, she reaches 2 before 0, and with probability $\frac{1}{2}$, she reaches 0 before 2. If she never returns to 0, then she reaches 2 before 0, with probability $\frac{1}{2}$. She is now equidistant from 0 and 4. Again, with probability $\frac{1}{2}$, she reaches 4 before 0. And so on; she reaches 2^{n+1} from position 2^n before 0 with probability $\frac{1}{2}$. So she never returns to 0 with probability $\frac{1}{2^n}$ as $n \to \infty$, which is just 0; as a result, she always returns home. It's easy to see that once she returns home, just run this argument again to conclude that she returns home infinitely often.

You may say: hold up. What if she just ambles about in space and never hits 0 or 2^n for some fixed n? It turns out that this is impossible - convince yourself!

If you think about this further, this isn't a very good solution, because it is hard to generalize to 2D lattices and above. So we'll give it a more mathematical treatment.

5 Math to the rescue

Instead of calculating the probability of returning, let's calculate the expected number of times that a random walk will return home. Let's suppose $\mathbb{E}$ is the expected number of returns to 0. If $\mathbb{E}$ is infinite, then we know that the probability is 1; if $\mathbb{E} < \infty$, then the probability is strictly between 0 and 1.

This deserves some discussion.

5.1 Translating between probability and expected value

If the probability is indeed 1, then she is mathematically guaranteed to return home in finitely many steps; then restarting the process, she will pick up arbitrarily many returns home. On the other hand, if the probability p is strictly less than 1, then she may return home, but the probability that she does so n times is p^n, and in particular the expected number of returns home is

$$\mathbb{E} = \sum_{n=1}^{\infty} p^n = \frac{1}{1-p} < \infty.$$

So we can associate $\mathbb{E} = \infty \leftrightarrow p = 1$ and $\mathbb{E} < \infty \leftrightarrow p < 1$. **This is how pretty much all of our arguments will go: we will calculate the expected number of returns, rather than directly calculating the probability of returning.**

5.2 Back to the story

Then the probability of returning at 0 after $2n$ turns is $\frac{\binom{2n}{n}}{2^{2n}}$: there are $\binom{2n}{n}$ valid sequences of -1 and $+1$ of length $2n$ which sum to 0. On the other hand, for the $2n$ steps, there are 2 equally likely outcomes. We now have to evaluate

$$\mathbb{E} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{2^{2n}}.$$

The primary result we will need is

Theorem 5.1 (Stirling's formula)
Asymptotically, $n! \sim \frac{n^n}{e^n}\sqrt{2\pi n}$.

(What this means is that as $n \to \infty$, the limit of $\frac{n!}{\frac{n^n}{e^n}\sqrt{2\pi n}}$ goes to 1. But you should think about it as: when n gets pretty large, the two expressions are about the same.)

Using Stirling's formula, we find that

Corollary 5.1.1. $\frac{\binom{2n}{n}}{2^{2n}} \sim \frac{1}{\sqrt{\pi n}}$.

We will use this result quite a bit!

So how do we actually solve this problem? Well, the expected value becomes approximately the sum of

$$\mathbb{E} \approx \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \to \infty.$$

Remark. This is of course not mathematically rigorous, but we'll skip over tedious details to make it more rigorous since it doesn't really highlight the main ideas. Of course you cannot simply say that two things are approximately equal, so an infinite sum of one is approximately an infinite sum of the other. However, one thing which is worth pointing out, is that partial sums can be made to be arbitrarily accurate by leaving the first N terms intact and only approximately the terms from $N+1$ onwards, i.e.

$$\sum_{n=1}^{N} \frac{\binom{2n}{n}}{2^{2n}} + \frac{1}{\sqrt{\pi}} \sum_{k=N+1}^{\infty} \frac{1}{\sqrt{k}}.$$

The important part is that the sum of $\frac{1}{\sqrt{k}}$ always diverges, no matter what N you pick (no matter where you start adding from).

So yes, Phyllis the ant does return home, and infinitely often. Yay!

6 A mouse on a grid

After returning home infinitely often, Phyllis undergoes rapid evolution and evolves into a mouse. Biology is incredible, huh?

She's still drunk, but she's no longer on a string. Now she lives on a flat 2D surface $\mathbb{Z}^2$. At every step, with probability each $\frac{1}{4}$, she will go one step north, one step south, one step east, or one step west.

Mathematically, let's say she starts at $(0,0)$ in $\mathbb{Z}^2$ and at each step, she moves from (m,n) to $(m+1,n)$, $(m,n+1)$, $(m-1,n)$, or $(m,n-1)$, each with probability $\frac{1}{4}$. Does she ever return home?

6.1 A naive attempt

Now, the probability that she returns home after $2n$ steps depends on how many steps she took in the x direction and how many steps she took in the y direction. Suppose she took $2k$ steps in the x direction. She would have $\binom{2k}{k}\binom{2n-2k}{n-k}\binom{2n}{2k}$ valid sequences. Then the probability that she returns home after $2n$ steps is

$$\frac{\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k}\binom{2n}{2k}}{4^{2n}}.$$

It is fairly straightforward, so I'll sketch it out here. But you can ignore this part and skip to the next section if you want. (This is for those who don't believe in combinatorics.)

We write out the value in terms of factorials:

$$\begin{aligned}
\binom{2k}{k}\binom{2n-2k}{n-k}\binom{2n}{2k} &= \frac{(2k)!}{k!k!}\cdot\frac{(2n-2k)!}{(n-k)!(n-k)!}\cdot\frac{(2n)!}{(2k)!(2n-2k)!},\\
&= \frac{(2n)!}{k!k!(n-k)!(n-k)!},\\
&= \frac{(2n)!}{n!n!}\cdot\frac{n!}{k!(n-k)!}\cdot\frac{n!}{k!(n-k)!},\\
&= \binom{2n}{n}\cdot\binom{n}{k}^2.
\end{aligned}$$

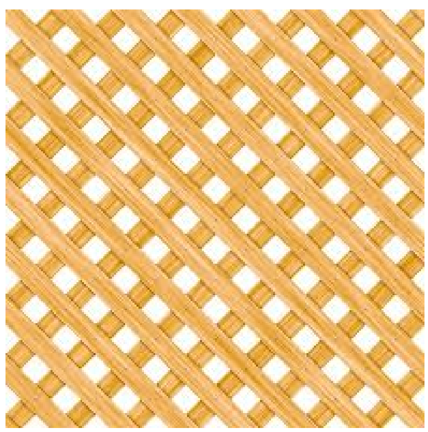
Then using the identity $\sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}$, we obtain that

$$\mathbb{E} = \sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{4^{2n}}.$$

Anyways, I'll try a different approach.

6.2 Tilt the lattice

The clever solution is simply to tilt the lattice 45°.



Let's say that she starts at $(0,0)$, and every step, she goes from (m,n) to one of $(m+1,n+1)$, $(m-1,n+1)$, $(m+1,n-1)$, $(m-1,n-1)$, each with probability $\frac{1}{4}$. In particular, her x and y coordinates are now independent of each other, and they each behave like a random walk on $\mathbb{Z}$! (Technically, those ± 1 should be $\pm 1/\sqrt{2}$, but scaling doesn't hurt anyone.)

It's now clear that the probability that she returns to the origin after $2n$ steps is merely the square of the probability in the $\mathbb{Z}$ case, and therefore is $\frac{\binom{2n}{n}^2}{4^{2n}}$. Thus the expected number of returns is

$$\mathbb{E} = \sum_{n=1}^{\infty}\frac{\binom{2n}{n}^2}{4^{2n}} \sim \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n} \to \infty.$$

It still diverges - so Phyllis the mouse always returns home, and infinitely often.

Now, if you've studied p-series convergence and divergence, you're surely noticing that we've reached the limit of divergence. You might have already guessed the next phenomenon!

7 A bird in the sky

Phyllis still returns infinitely often, so she ends up evolving into a bird. She can now fly, but there's no ground, or upper limit to the sky, or...

Let's say she starts at $(0,0,0)$ in $\mathbb{Z}^3$. At each step, she moves from (ℓ, m, n) to one of $(\ell+1, m, n)$, $(\ell-1, m, n)$, $(\ell, m+1, n)$, $(\ell, m-1, n)$, $(\ell, m, n+1)$, or $(\ell, m, n-1)$, each with probability $\frac{1}{8}$. Using the tilting idea from before, this is the same as moving to $(\ell \pm 1, m \pm 1, n \pm 1)$, where each $\pm$ is independent.

Now the probability of returning home after $2n$ steps is $\left(\frac{\binom{2n}{n}}{2^{2n}}\right)^3$, and as a result, her expected number of returns home is

$$\mathbb{E} = \sum_{n=1}^{\infty} \left(\frac{\binom{2n}{n}}{2^{2n}}\right)^3 \sim \frac{1}{\pi\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty.$$

This converges! We cannot say the exact value, since we used some gross approximations that throw off the initial terms really badly. However, those approximations are accurate in the distant terms with large n, and those terms determine convergence or divergence. So this sum indeed converges, which implies that $p < 1$. (Wikipedia says that $\mathbb{E} \approx 1.516...$ and actually there is a closed form using the Γ function!)

So a drunk bird may never return home. What a shame! Aren't you glad you walk in two dimensions?